

Exact Energy Levels and Eigenfunctions of an Electron on a Nanosphere Under the Radial Magnetic Field

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The exact energy levels and wave functions of an electron free to move on a sphere under the radial magnetic field is found. Wave functions are expressed in terms of Jacobi polynomials which were well-defined and have orthogonality relation, recurrence relations, series expansions etc. We also discussed the the wave functions and energy levels in case of very large magnetic field. Landau energy levels are shown for strong constant magnetic fields occurring on two-dimensional flat surfaces, if the radius is very large. The results compared with those previously found in the literature.

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I. INTRODUCTION

Flat, two-dimensional systems are carried out on helium and quantum wells. Similarly, curved two-dimensional systems can be created by both on helium and solid state systems. In the first case, the multi-electron bubble in the liquid helium and the second example is the metallic nanoshell [1].

The electronic properties of curvilinear surfaces have attracted much interest since the discovery of carbon nanostructures. The motion of charged particles on a sphere under constant magnetic field have been studied by authors for discuss the fractional quantum Hall Effect [2–5], weak magnetic field and strong magnetic field properties [6, 7], Landau Levels [8, 9] and optical properties [10].

Goddart and Olive [11] gave a detailed description of the methods of creating radial magnetic fields (Dirac magnetic monopole and Dirac string). The energy levels of the charged particle on a sphere under a radial magnetic field was investigated by Ralko and Truong in classical and quantum mechanic regimes. They obtained solutions in Heun Functions, which is a generalization of the Gauss Hypergeometric Function. In one of the coefficients of the Heun equation, they found a condition that led to the quantification of energy levels [12].

While Haldane [2], Ralko and Truong [12] determine the single particle wave functions, they take the magnetic vector potential as $\vec{A} = -\frac{\hbar S}{eR} \cot \theta \hat{\phi}$ due to the radial magnetic field. In this study, the magnetic vector potential determined in an efficient manner in section 2. It has been assumed that there is no radial dependence of the movement of the electron on the sphere and it depends on only two angle variables. By exactly solving the time independent Schrödinger equation, we found the exact energy eigenvalues and the wave functions of an electron on a two-dimensional spherical surface under a radial magnetic field. The eigenfunctions have been expressed in terms of Jacobi polynomials that has orthogonality and recurrence relations. In the absence of magnetic field, it has been shown that eigenfunctions are reduced to Legendre polynomials. In the $p \rightarrow \infty$ limit, it is shown eigenfunctions are reduced to Laguerre polynomials. Furthermore, the limit $R \rightarrow \infty$ is discussed, in which case the sphere surface can be taken as a flat surface, and energy levels have been shown to turn into two-dimensional Landau energy levels. In the last part, the results have been discussed.

II. NANOSPHERE UNDER THE RADIAL MAGNETIC FIELD

The electrons on a sphere are strongly bound perpendicular to the surface, while they move freely in directions parallel to the surface. On a sphere, this means that the full (three-dimensional) wave function describing such electrons should be factorizable in a function depending only on the angles and a function depending only on the radial distance. The system can be considered two-dimensional if all the electrons have the same radial dependence of their wave function, and if the energy required to change the radial mode is much larger than the other relevant energy scales involved.

Electrons that can move freely on the sphere are in single-particle angular momentum states. For a rigid sphere of radius R , the energy of a single electron confined to the surface is called the rigid rotator energy $E_\ell = \frac{\hbar^2}{2m^* R^2} \ell(\ell + 1)$, where ℓ is the angular momentum quantum number, m^* the mass of the electron, and \hbar is Planck's constant.

The Schrödinger equation of an electron on a metallic nanosphere under the radial magnetic field

$$\frac{1}{2m^*} \left[-i\hbar\nabla + \frac{e}{c}\mathbf{A} \right]^2 \psi = E\psi \quad (1)$$

Where \mathbf{A} is the vector potential on the sphere due to the radial magnetic field, m^* and e effective mass and charge of the electron respectively and c is speed of light.

Using spherical polar coordinates (r, θ, ϕ) we expect by symmetry to be able to find a vector potential $\mathbf{A}(\theta) = A(\theta) \hat{\phi}$, $\hat{\phi}$ being a unit vector in the ϕ direction. The magnetic flux through a circle, C , corresponding to fixed values of R and θ , and ϕ ranging over the values 0 to 2π , is given by the solid angle subtended by C at the origin multiplied by $\frac{1}{4\pi} \int \mathbf{B} \cdot d\mathbf{S}$, namely $\frac{(1-\cos\theta)}{2} \int \mathbf{B} \cdot d\mathbf{S}$. The total magnetic flux from the sphere surface is $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$ [11]. Consequently:

$$\frac{(1-\cos\theta)}{2} \Phi = 2\pi A(\theta) R \sin\theta \quad (2)$$

and the vector potential

$$\mathbf{A}(\theta) = \frac{\Phi}{4\pi R} \frac{(1-\cos\theta)}{\sin\theta} \hat{\phi} \quad (3)$$

if we rewrite the vector potential in the Eq. (3) into the Eq(1) and look for the eigenfunctions in the form $\psi(\theta, \phi) = T(\theta) e^{im\phi}$

$$-\frac{\hbar^2}{2m^*R^2} \left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dT}{d\theta} \right) - \frac{m^2}{\sin^2\theta} - mp \frac{1-\cos\theta}{\sin^2\theta} - \frac{p^2}{4} \frac{(1-\cos\theta)^2}{\sin^2\theta} \right] T(\theta) = E T(\theta) \quad (4)$$

where $m\hbar$ is the eigenvalue of \mathbf{L}_z , magnetic flux quantum $\Phi_0 = \frac{hc}{e}$ and $\Phi/\Phi_0 = p$ denotes the magnetic flux in unit of the flux quantum Φ_0 . Defining dimensionless energy $\epsilon = \frac{2m^*R^2}{\hbar^2} E$ and changing the variable $\mu = \cos\theta$, we write

$$\frac{d}{d\mu} \left[(1-\mu^2) \frac{dT(\mu)}{d\mu} \right] + \left[\epsilon - \frac{m^2}{1-\mu^2} - mp \frac{1-\mu}{1-\mu^2} - \frac{p^2}{4} \frac{(1-\mu)^2}{1-\mu^2} \right] T(\mu) = 0 \quad (5)$$

Let us define the $T(\mu)$ wave function as:

$$T(\mu) = (1-\mu)^{\frac{|m|}{2}} (1+\mu)^{\frac{p+m}{2}} P(\mu) \quad (6)$$

where $p+m \geq 0$ and m can take $-p \leq m \leq p$ integer values. After substituting Eq. (6) into Eq. (5), we find

$$(1-\mu^2) \frac{d^2 P(\mu)}{d\mu^2} + [p+m-|m| - (p+m+|m|+2)\mu] \frac{dP(\mu)}{d\mu} + \left(\epsilon - \frac{(p+m)(m+|m|+1)+|m|}{2} \right) P(\mu) = 0 \quad (7)$$

this differential equation is Jacobi Differential Equation and its solution are Jacobi Polynomials, $P_\ell^{(|m|, p+m)}(\mu)$ and ℓ is angular momentum quantum number that nonnegative integer. To find the energy eigenvalues, we use dimensionless energy identity $\epsilon = \ell(\ell+1) + |m|(\ell+\frac{1}{2}) + (p+m)\left(\ell + \frac{m+|m|+1}{2}\right)$

$$E_{\ell,p,m} = \frac{\hbar^2}{2m^*R^2} \left[\ell(\ell+1) + |m|(\ell+\frac{1}{2}) + (p+m) \left(\ell + \frac{m+|m|+1}{2} \right) \right] \quad (8)$$

This energy eigenvalue is the energy eigenvalue of an electron that freely moves on the surface of a sphere under the radial magnetic field. And total wave function

$$\psi(\mu, \phi) = N_\ell^{p,m} (1-\mu)^{\frac{|m|}{2}} (1+\mu)^{\frac{p+m}{2}} P_\ell^{(|m|, p+m)}(\mu) e^{im\phi} \quad (9)$$

from the orthogonality property of Jacobi Polynomials, the normalization constant $N_\ell^{p,m}$ given as:

$$N_\ell^{p,m} = \left[\ell! \frac{(2\ell + p + m + |m| + 1)}{\pi} \frac{\Gamma(\ell + p + m + |m| + 1)}{2^{p+m+|m|+2} \Gamma(\ell + |m| + 1) \Gamma(\ell + p + m + 1)} \right]^{1/2} \quad (10)$$

where $\Gamma(z)$ the Gamma Function. In the absence of a magnetic field, the wave function is reduced to $\psi(\mu, \phi) = \frac{2\ell+1}{4\pi} P_\ell(\mu) e^{im\phi}$ where $P_\ell(\mu)$ Legendre polynomials.

Now, we examine $p \rightarrow \infty$ ($p + m \rightarrow \infty$) limit. If we change the variable μ to $x = \frac{p+m}{2}(1 - \mu)$, functions in Eq.(9) in this limit $\lim_{p+m \rightarrow \infty} P_\ell^{(|m|, p+m)}(1 - \frac{2x}{p+m}) = L_\ell^{|m|}(x)$ [13] where $L_\ell^{|m|}(x)$ Associated Laguerre Polynomials, $(1 - \mu)^{\frac{|m|}{2}} = \left(\frac{2}{p+m}\right)^{\frac{|m|}{2}} x^{\frac{|m|}{2}}$ and $\lim_{p+m \rightarrow \infty} (1 + \mu)^{\frac{p+m}{2}} \simeq 2^{\frac{p+m}{2}} e^{-\frac{x}{2}}$. Then the wave function becomes

$$\psi(x, \phi) = N_{\ell,m} e^{-\frac{x}{2}} x^{\frac{|m|}{2}} L_\ell^{|m|}(x) e^{im\phi} \quad (11)$$

normalization constant $N_{\ell,m}$ is found from the orthogonality property of Associated Laguerre Polynomials $N_{\ell,m} = \left(\frac{\ell!}{2\pi (\ell + |m|)!} \right)^{\frac{1}{2}}$.

III. THE $R \rightarrow \infty$ LIMIT

In this section, we will observe that the wave function and energy belong to two-dimensional spherical surface under radial magnetic field are converted to the wave function and energy of an electron under the constant magnetic field in the z direction on two-dimensional flat surfaces.

In the case of small θ we can use plane polar coordinates in Eq. (4), if we take $\sin\theta = \rho/R$ and $z \simeq R$, $\cos\theta = 1 - \rho^2/2R^2$ and we take the wave function in the form $\psi(\rho, \phi) = T(\rho) e^{im\phi}$

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dT(\rho)}{d\rho} \right) + \left(\varepsilon - \frac{m^2}{\rho^2} - \frac{p^2}{16R^4} \rho^2 \right) T(\rho) = 0 \quad (12)$$

where $\varepsilon = \frac{2m^*E}{\hbar^2} - \frac{m}{2R^2}$. After change of variable $x = \sqrt{\frac{p}{4R^2}} \rho$, we arrive at

$$\frac{1}{x} \frac{d}{dx} \left(x \frac{dT(x)}{dx} \right) + \left(\lambda - \frac{m^2}{x^2} - x^2 \right) T(x) = 0 \quad (13)$$

where $\lambda = \frac{4R^2}{p} \varepsilon$ and considering the limit states ($x \rightarrow 0$ and $x \rightarrow \infty$) of Eq. (13), we can propose the solution as

$$T(x) = e^{-x^2/2} x^{|m|} L(x) \quad (14)$$

If we substitute Eq. (14) into Eq. (13) and then changing the variable $y = x^2$, we arrive at

$$y \frac{d^2 L(y)}{dy^2} + (|m| + 1 - y) \frac{dL(y)}{dy} + \left(\frac{\lambda}{4} - \frac{|m| + 1}{2} \right) L(y) = 0 \quad (15)$$

This differential equation is Associated Laguerre differential equation and its solutions are Associated Laguerre polynomials. To find energy eigenvalues, $\frac{\lambda}{4} - \frac{|m|+1}{2} = n$ can be used where n is an integer ($n = 0, 1, 2, \dots$). If we use total magnetic flux $\Phi = \int \mathbf{B} \cdot d\mathbf{S} = 4\pi R^2 B$

$$E_{n,m} = \hbar\omega_c \left(n + \frac{m + |m| + 1}{2} \right) \quad (16)$$

where $\omega_c = eB/m^*c$ cyclotron frequency and B constant magnetic field in z direction on the flat surface. Energy levels in Eq. (16) is the well-known Landau energy levels [14].

IV. DISCUSSIONS

In this work, we have exactly solved the Schrödinger equation of a single particle at the spherical surface and under the radial magnetic field to find electronic energy levels and wave functions. To analyze the energy levels we have found, consider Eq. (8). In the absence of the magnetic field ($p = 0$ and $m = 0$), these energy levels are reduced to $E_{\ell,0,0} = \frac{\hbar^2}{2m^*R^2} \ell(\ell+1)$ rigid rotator energy levels as expected. For fixed values of ℓ and p in Eq. (8), the difference between energy levels for positive m is $\Delta E = E_{\ell,p,m+1} - E_{\ell,p,m} = \frac{\hbar^2}{2m^*R^2} (2\ell + p + 2m + 2)$. Although the difference between the Landau energy levels $\Delta E = \frac{1}{2}\hbar\omega_c$ for positive m in flat surfaces is equidistant, the difference between the energy found for curved surfaces is $\Delta E = E_{\ell,p,m+1} - E_{\ell,p,m} = \frac{\hbar^2}{2m^*R^2} (2\ell + p + 2m + 2)$ depend on m and is not equidistant.

In Fig. (1), we plot energy levels in Eq. (8) in $\left(\frac{\hbar^2}{2m^*R^2}\right)$ unit as a function of m (magnetic quantum number) and ℓ (angular momentum quantum number) for fixed value of $p = 10$. For negative m values, these energy values are not depend on m , but for positive m values the energy is increased by increasing m . Energy eigenvalues increase with increasing ℓ (angular momentum quantum number) values. In Fig. (1), energy levels for $\ell = 0$, $\ell = 5$ and $\ell = 10$ denoted by dots, diamonds and triangles, respectively.

In section 3, it is shown that the energy levels of an electron on the spherical surface under the radial magnetic field turn into Landau energy levels at the $R \rightarrow \infty$ limit. Landau energy levels are not dependent on m (magnetic quantum number) for negative m values but increase linearly with increasing positive m values as seen from Eq. (16). The energy levels of an electron on a spherical surface under the radial magnetic field are similar to those of the Landau energy levels but there are differences due to spherical surface effects at the $R \rightarrow \infty$ limit, these spherical surface effects cease to exist.

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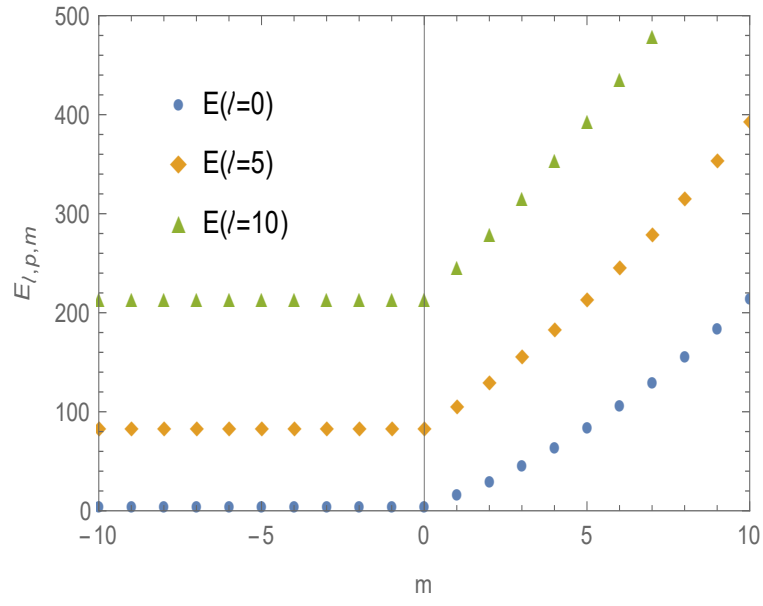


FIG. 1: Energy levels of an electron on a sphere under the radial magnetic field as a function of magnetic quantum number m and angular momentum quantum number ℓ for special value of magnetic field ($p=10$)